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Fault Tolerant Bayesian Implementation in Exchange Economies*

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ABSTRACT. In this work, we extend the concept of Fault Tolerant Implementation of Eliaz (2002) to the concept of Fault Tolerant Implementation in incomplete information environments. In particular, we work in a domain where information is non-exclusive by choosing a model of pure exchange economy. As in Eliaz (2002), we suppose the existence of at most k faulty players who do not act in an optimal way, either because they do not understand the rules of the game or they make mistakes. We develop a new concept of equilibrium, called k-Fault Tolerant Bayesian Equilibrium ($k-FTBE$) and a new concept of implementation, called fault tolerant Bayesian implementation. In model of pure exchange economy, we show that weak k- Bayesian monotonicity is a necessary condition for the implementation of social choice correspondences in $k-FTBE$. We also introduce the no-exclusiveness information condition ($k-NEI$), and we show that k-Bayesian monotonicity and $k-NEI$ are sufficient conditions for implementation when there are at least three players and the number of the faulty players is less then $\frac{1}{2}n-1$.

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1 Introduction

Because of increased criticism with regard to the confidence of the classical implementation theory on full rationality, Eliaz, in a recent paper (Review of Economic Studies (2002)) appears to have tried to provide a partial answer to this criticism. It represents an attempt to incorporate a model of bounded rationality in the implementation theory.

The standard approach to implementation supposes implicitly that each agent can choose correctly its most preferred strategy. A question arises on the robustness of the standard models when there are slight deviations of full rationality.

Eliaz (2002) supposes that all players are not always rational in their behavior. It may be that there exists a certain number of players who are faulty in the sense that they do not act in an optimal way because they do not understand the rules of the game or make mistakes. The planner and the non-faulty players know only that there are at most k faulty players in the population. However, they know neither the identity of the faulty players nor their exact number nor their behavior. Eliaz defined a new concept of equilibrium, called k-Fault Tolerant Nash Equilibrium ($k - FTNE$). He showed that weak k-monotonicity is a necessary condition and that k-monotonicity together the k-no veto power condition are sufficient conditions for implementation in $k - FTNE$ when there are at least three players and the number of faulty players is less than $\frac{1}{2}n - 1$.

In our work, we extend the concept of this literature in the environments of incomplete information. We will try at the beginning to define a new concept of Bayesian equilibrium when there exists at most k faulty players. Our equilibrium will be called, k-Fault Tolerant Bayesian Equilibrium ($k - FTBE$). Next, we will define a new concept of implementation in this equilibrium.

Generally, in environments of incomplete information, it is important to distinguish between exclusive and non-exclusive information. If information is exclusive, it is impossible to detect all false states on the part of individuals. If on the other hand information is nonexclusive, no agent has exclusive private information, i.e., the information of each agent is superfluous in the sense that it is implied by the collective information of the other agents. In this case, therefore there are no incentive compatibility constraints.

Postelwaite and Schmeidler (1986), and Palfrey and Srivastava (1987) characterize implementable social choice rules in the exchange economies in which information is not exclusive. Palfrey and Srivastava (1989) examine implementation in exchange economies for which the players can have exclusive information. Jackson (1991) extends these results with either the exclusive or non-exclusive information in general economic environments.

When information is nonexclusive, the Bayesian monotonicity condition is necessary and sufficient for the implementation in exchange economies when there are at least three players. On the other hand, when information is exclusive, the Bayesian monotonicity condition is insufficient for implementation, but it remains necessary. Palfrey and Srivastava (1989) show that Bayesian monotonicity and incentive compatibility are two necessary conditions for Bayesian implementation in exchange economies and they become sufficient with strong incentive compatibility.

In this work, we will construct a model adapted from those of Postelwaite and Schmeidler (1986), and Palfrey and Srivastava (1987). The information among the players is asymmetrical and nonexclusive. We consider the existence at most k faulty players who are nonmajority among the set of the players. The number and the identity of these faulty players are unknown either by the planner or by the non-faulty players. It will be a natural extension from the weak k-monotonicity and k- monotonicity of Eliaz to the weak k- Bayesian monotonicity and k- Bayesian monotonicity. We show that in the nonexclusive information environment, k-Bayesian monotonicity is sufficient for the implementation in $k - FTBE$ when there are at least three players and the number of the players faulty is less than $\frac{1}{2}n - 1$ and that the weak k-Bayesian monotonicity is a necessary condition.

The paper is organized as follows. The model and definitions are laid out in Section 2. In section 3 and 4, we define new concepts of equilibrium and implementation. Section 5 establishes the necessary an sufficient conditions for fault tolerant Bayesian implementation. We close with concluding remarks.

2 Model and Definitions

Let E be a pure exchange economy in incomplete information environments represented by the list: $\{N, S, (\Pi^i, U^i, w^i, q^i(.,.))_{i\in N}\}$, where
$N = \{1, ..., n\}$ is the set of agents in an exchange economy.
$S = \{s_1, s_2, ..., s_r\}$ is a finite set of states, each $s \in S$ describes the set of agents, their endowments, and their preferences. We assume that the number of agents and the aggregate endowment are independent of the state.
Π^i: is a partition of S which represents the structure of information of an agent i, the elements of Π^i are called: *the events*, each event $E^i \in \Pi^i$ is a maximal set of states that agent i cannot distinguish. If the state is s, we suppose that the agent i knows only that the true state lies in a set $E^i(s) \subset S$.
$U^i : \Re^l_+ \times S \longmapsto \Re$ is the utility function of agent i in state s, and is assumed to be strictly increasing and bounded below for each s. We normalize $U^i(0, s) = 0$ for all i and s.
$w^i(s) : S \longmapsto \Re^l_+$: is a function that represents the initial allocation of the agent i. The initial endowments are elements of the non-negative consumption set $\Re^l_+$ of the Euclidean space of dimension l, such that l represents the Arrow-Debreu commodities in the economy. Consumption sets are the nonnegative orthant.

According to the neobayesian paradigm, every economic agent has a (prior) probability distribution over S defined by q^i, and we assume that $q^i(s) > 0$. The conditional (posterior) probability is given by

$$q^i(t|E^i(s)) = \begin{cases} 0 & \text{if} \quad t \notin E^i(s), \\ \frac{q^i(t)}{q^i(E^i(s))} & \text{if} \quad t \in E^i(s). \end{cases}$$

$q^i(t|E^i(s))$: means that player i knows his own event, but does not know the events of the other players. He evaluates the probabilities of these other players that having several configurations of the events; these evaluations are recapitulated by the measurement of probability q^i.

In complete information environments, an allocation is a distribution of the aggregate endowments among the agents, i.e., a profile $a = (a_i)_{i\in N}$ where $a_i \in \Re^l_+$ for all $i \in N$. The feasible allocations set is denoted by

$$A = \left\{ a \in \Re^l_+ \mid \sum_{i\in N} a_i \leq \sum_{i\in N} w_i \right\}$$

Since states are not known, an allocation for agent i is not a point in A, but *a social choice function* $x_i : S \to \Re^l_+$ that associates each state of the worlds a commodity vector . An allocation is also called *an allocation rule*. The feasible allocations set is given by

$$\mathscr{A} = \left\{ x \in \Re^l_+ \mid \sum_{i \in N} x_i \leq \bar{w} \right\}$$

where $\bar{w}$ represents the aggregate endowments, $\bar{w} \gg 0$.

Let $X = \{x : S \rightarrow \mathscr{A}\}$ the set of all social choice functions. *A social choice set* is a subset $F \subset X$. In other words, a social choice set is a collection of mapping from a set of states to a feasible allocation set.
The concept of a social choice set differs from that of a social choice correspondence, i.e., a correspondence which associates the states to allocations. But, if the condition of closure [1] is satisfied, the two concepts become equivalent.

We define a *faulty* player as Eliaz (2002). A player will be called faulty if he does not act according to incentives. That is, given the strategies chosen by the other players, a faulty player does not choose a strategy that leads to his most preferred outcome because he does not understand the rules of the game or he makes errors.

We suppose that there exists in a population N at most k faulty players such that k is a fixed number. Any subset of these players might play in an unpredictable manner. A non-faulty player knows that he is non-faulty. But he cannot tell whether another player is faulty or not, and he does not know the exact number of faulty players in N. He only knows that there cannot be more than k faulty players in N. A faulty player does not know that it is faulty. The planner only knows that there can be at most k faulty players. He cannot distinguish among the faulty players and the non-faulty players in N.

In this work, in addition to uncertainty on the states of nature, we introduce uncertainty on the nature of the players.

In our incomplete information environment, we suppose that if a player is faulty, he stays faulty for all states of nature $s \in S$. Let $p^i : N \longmapsto [0, 1]$ be a distribution of conditional probability on nature of the players (faulty or non-faulty).

Let M be a random subset of agents such that $M \subseteq N$. Let $\widetilde{M}$ be a random subset of faulty players such that $\widetilde{M} \subseteq N$ and $\mid \widetilde{M} \mid = \widetilde{k}$ with $\widetilde{k} \leq k$. Let $\widetilde{N} = N \setminus \widetilde{M}$ a random subset of non-faulty players.

[1] See definition in condition 2

Each player believes that he is not faulty and he gives a probability to each other player to be faulty. Formally, this probability is measured by[2]

$$p^i(j\ is\ faulty|i\ is\ non-faulty) = \begin{cases} \frac{\sum_{\widetilde{M},|\widetilde{M}|\leq k} p(\widetilde{M}|j\in\widetilde{M}, i\notin\widetilde{M})}{\sum_{\widetilde{M},|\widetilde{M}|\leq k} p(\widetilde{M}|i\notin\widetilde{M})} & if\quad k>0, \\ 0 & if\quad k=0. \end{cases}$$

Because we need a concept of solution as well as results of a mechanism robust to deviations of faulty players, we define the conditional probability of a player i on the nature of the other players to be non-faulty by,

$$p^i((j\neq i)\in\widetilde{N}|i\in\widetilde{N}) = \begin{cases} 1-\frac{\sum_{\widetilde{M},|\widetilde{M}|\leq k} p(\widetilde{M}|j\in\widetilde{M}, i\notin\widetilde{M})}{\sum_{\widetilde{M},|\widetilde{M}|\leq k} p(\widetilde{M}|i\notin\widetilde{M})} & if\quad k>0, \\ 1 & if\quad k=0. \end{cases}$$

where $\widetilde{N}$ and $\widetilde{M}$ are events, in other words, random variables. A non-faulty player knows the existence of an event which is constituted of at least $n-k$ players, he knows also its membership at this event, but, he does not know its members.

$p^i((j\neq i)\in\widetilde{N}|\widetilde{N}, i\in\widetilde{N})$: means that player i knows the event to which it belongs, but does not know the memberships of the other players. Therefore, he evaluates the probabilities of these other players that having two events; these evaluations are summarized by an measure probability p^i.

A player $i\in N$ evaluates the expected utility of x_i after he uses its probability on nature of the other players and its probability on the states of nature of the other players by using the Bayes rule. The expected utility of allocation x_i of a player i in state s with $|\ \widetilde{M}\ |\leq k$ is,

$$V_i^k(x,s) = \sum_{t\in E^i(s)} p^i(j\neq i\in\widetilde{N}|\widetilde{N}, i\in\widetilde{N})q^i(t\mid E^i(s))U^i(x_i(t),t).$$

The preference relation R^i is defined on X by $xR^i(E^i(s))y$ if and only if,

$$\sum_{t\in E^i(s)} p^i(j\neq i\in\widetilde{N}|\widetilde{N}, i\in\widetilde{N})q^i(t\mid E^i(s))U^i(x_i(t),t)$$

[2]For example, suppose that $N=\{1,2,...,7\}$ and $k=2$. Suppose also that $i=2$ is non-faulty and $j=1$ is faulty. Therefore, the random subset of faulty players $\widetilde{M}=$ $\{1\},\{3\},\{4\},\{5\},\{6\},\{7\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{1,7\},\{3,4\},\{3,5\},\{3,6\},\{3,7\}\{4,5\},$ $\{4,6\},\{4,7\},\{5,6\},\{5,7\},\{6,7\}$.Thus, $p^i(j=1\ is\ faulty|i=2\ is\ non-faulty)=\frac{6}{21}$.

$$\geq$$

$$\sum_{t\in E^i(s)} p^i(j \neq i \in \widetilde{N} | \widetilde{N}, i \in \widetilde{N}) q^i(t \mid E^i(s)) U^i(y_i(t), t).$$

The asymmetrical and symmetrical parts of $R^i(E^i(s))$ are noted respectively by $P^i(E^i(s))$ and $I^i(E^i(s))$.

If $k = 0$, i.e., if all players are non-faulty, then the probability $p^i(j \neq i \in \widetilde{N} | \widetilde{N}, i \in \widetilde{N}) = 1$.

Thus the expected utility $V_i^0(x, s)$ becomes equivalent to classic expected utility

$$V_i(x, s) = \sum_{t\in E^i(s)} q^i(t \mid E^i(s)) U^i(x_i(t), t).$$

For the implementation results, we need to put more structure on information in order to simplify the model. Like Postelwaite and Schmeidler (1986), and also Palfrey and Srivastava (1987, 1989), we give the following condition,

Assumption 1 (No Redundant States) .
$\forall M \subseteq N, \quad \mid M \mid \geq k+1, \quad \forall i \in M, \quad \forall s \in S, \quad \bigcap_{i\in M} E^i(s) = \{s\}.$

This condition ensures that any appropriate private information is taken by some agent in the economy.

Assumption 2 (Closure) .
Denote by $\Pi^* = \bigwedge_{i\in N} \Pi^i$ the finest partition of S which is coarser than Π^i for every $i \in N$. We also call Π^* the *common knowledge partition* of S. The element $\Pi^*(s)$ of Π^* containing s is *common knowledge at* s. An event E is said to be common knowledge at s if $\Pi^*(s) \subset E$.

We define y to be the common knowledge concatenation of x_1 and x_2 if $y(t) = x_1(t)$ for $t \in E$ and $y(t) = x_2(t)$ for $t \notin E$. If for any $x_1, x_2 \in F$, for any $E \in \Pi^*$, the common knowledge concatenation y has the property $y \in F$, then F is said to satisfy *closure*.

Definition 1 (Mechanism) *:*
A mechanism Γ *is an action space* $\mathscr{M} = \mathscr{M}^1 \times \mathscr{M}^2 ... \times \mathscr{M}^n$*, and a function* $g : \mathscr{M} \longrightarrow A$*. If* $\mathscr{M}^i = \Pi^i$ *for all* i*, then* $(\mathscr{M}, g)$ *is direct mechanism.*

Definition 2 (Pure strategy) *:*
A pure strategy for an agent i *is a mapping* $\sigma^i : \Pi^i \rightarrow \mathscr{M}^i$.

For any state s, let σ be a vector of strategies such that $\sigma(E(s)) = (\sigma^1 E^1(s), ..., \sigma^n(E^n(s)))$; $\sigma = (\sigma^1, ..., \sigma^n); \sigma^{-i} = (\sigma^1, ..., \sigma^{i-1}, \sigma^{i+1}, ..., \sigma^n)$ and $g(\sigma) = (g(\sigma(E(s_1))), ..., g(\sigma(E(s_r))))$. The function $g(\sigma)$ represent the social choice function which results if σ is played.

Definition 3 (Deception) *:*
A deception for an agent i is a function $\alpha^i : \Pi^i \to \Pi^i$.

The name deception is derived from the fact that if α^i is interpreted as a strategy in a revelation mechanism, it indicates the event announced by i a function of the true event. The set of all deceptions for an individual i is equivalent to the set of all available and possible strategies for i in a direct mechanism, where the strategy which indicates the truth is quite simply the identity deception.

Suppose that in a direct mechanism, each agent use deception α^i, and at state s, each individual i reports $\alpha^i(E^i(s))$ instead of $E^i(s)$. Following Palfrey and Srivastava (1989), for each state s, there are two possibilities:
i) $\bigcap_i \alpha^i(E^i(s)) \neq \emptyset$: In this case, the intersection is a singleton. Thus, the reports of the agents are called *compatible*. In this case, the planner cannot tell whether anyone is lying.
ii) $\bigcap_i \alpha^i(E^i(s)) = \emptyset$: Means that the reports are *incompatible*. In this case, the planner can infer that some agent must be lying about his event.

In general, some group of deceptions will lead to compatible reports and some will not. Thus, the planner can prove effective incentives to prevent any equilibrium which involves incompatible reports. This allows us to restrict attention to compatible reporting strategies. Formally, we have the following definition of compatibility:

Definition 4 . *A group of deceptions $\alpha = (\alpha^1, ..., \alpha^n)$, with $\alpha^i : \Pi^i \to \Pi^i$, is compatible with Π if for all $(E^1, ..., E^n)$ such that $E^i \in \Pi^i$ for all i, $\bigcap_i E^i \neq \emptyset \Rightarrow \bigcap_i \alpha^i(E^i) \neq \emptyset$.*

From assumption 1 and definition 4, we conclude that: $\forall M \subseteq N, \mid M \mid \geq k+1, \forall i \in M, \forall s \in S$, $\alpha(s) = \bigcap_{i \in M} \alpha^i(E^i(s))$ for all α compatible with Π. We also define that: $x_\alpha(t) = x(\alpha(t)) = x \circ \alpha(t)$, $x_\alpha = (x_\alpha(s_1), ..., x_\alpha(s_r))$.

3 k-Fault Tolerant Bayesian Equilibrium (k-FTBE)

A equilibrium can at best describe a certain mode of stable behavior by the players who are potentially non-faulty. The planner must take into account that the players who prove to be faulty behave in an unpredictable manner. These players can choose a strategy contrary to their incentives.

Definition 5 (k-FTBE) *: A profile of strategies $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ is a k-FTBE, if for all $s \in S$, if no non-faulty player has incentive to deviate unilaterally from his equilibrium strategy σ_i^*, independently of the identity and the strategies of the faulty players as long as there exists n-k-1 non-faulty players who continue to play their equilibrium strategy $\sigma^*_{N\setminus(\{i\}\cup M)}$. Consequently, for all $i \in N$, we must to have*

$$g(\sigma_i^*, \sigma^*_{N\setminus M\cup\{i\}}, \sigma_M) R^i(E^i(s)) g(\sigma_i, \sigma^*_{N\setminus M\cup\{i\}}, \sigma_M),$$
$$\forall \sigma_i : \Pi^i \to \mathscr{M}^i, \forall \sigma_M : \Pi^M \to \mathscr{M}^M, \quad \forall M \subseteq N \ \ s.t. \ \mid M \mid \leq k$$

if $k = 0$, then $0 - FTBE$ is a Bayesian equilibrium.

We denote the set of $k - FTBE$ of the game $(\mathscr{M}, g)$ by $B^k(s, \mathscr{M}, g)$.

4 Fault Implementation

We define the set of the strategies which are different at equilibrium strategies, and which are able to be played by at most k-faulty players by

$$B(\sigma^*, k) = \{\sigma' : \Pi \to \mathscr{M} :\mid \{i \in N : \sigma_i^* \neq \sigma_i'\} \mid \leq k\}.$$

We consider full implementation for non-faulty players, which requires that the sets of the equilibrium outcomes of the mechanism for these players exactly coincide with the given social choice set. This does not allow the existence of any undesirable equilibrium in the mechanism.

Definition 6 *:*
In an economy, a social choice set is (full) implementable in k-FTBE if there is a mechanism $\Gamma = (\mathscr{M}, g)$ such that:
i) For any $x^* \in F$, there exists a k-FTBE σ^* for $(\mathscr{M}, g)$ such that $g(\sigma^*) = x^*$,
ii) If σ^* is a k-FTBE for $(\mathscr{M}, g)$, then $g(\sigma^*) \in F$,
iii) For each $\sigma^* \in B^k(s, \mathscr{M}, g)$, $g(B(\sigma^*, k)) \in F$.

A social choice set F is implementable in $k - FTBE$ if there exists a mechanism $(\mathscr{M}, g)$ which implements F in this equilibrium.

5 Necessary and sufficient conditions

In this section, we present the necessary and sufficient conditions that characterize the social choice set that can be implemented in $k - FTBE$, as long as the number of faulty players is not a majority.

5.1 Necessity

We begin by the following definition,

Definition 7 *(Weak k-Bayesian monotonicity):*
A social choice set F is weakly k-Bayesian monotonic if for any deception α compatible with Π, if whenever $x^* \in F$ and $x^*_\alpha \notin F$, $\quad \exists M \subseteq N, \quad | M | \geq k+1$ et $\exists y : S \rightarrow \mathscr{A}$ such that $\forall i \in M$, there exists an allocation $x^i \in F$ that satisfy $x^i R^i(\alpha(E^i(s)))y$, and for at least one player $j \in M, \quad y_\alpha P^j(E^j(s))x^j_\alpha$. [3]

In some mechanism, if an equilibrium outcome lie in a social choice set and this equilibrium outcome, generated by the group deception, do not, then there are at least $k+1$ players, each of whom previously considered one of the chosen outcomes to be at least as good as some given outcome, but at least one of these players reversed this relation.

Theorem 1 .
If a social choice set F is implementable in k-FTBE, then F satisfies weak k-Bayesian monotonicity.

Proof. Let F be a social choice set $k-FTBE$ implementable and $x^* \in F$. Thus, there exists a mechanism $\Gamma = (\mathscr{M}, g)$ that implements it and there exists a $k-FTBE$ $\sigma^* \in B^k(s, \mathscr{M}, g)$ with $g(\sigma^*) = x^*$,

We suppose that for some deception α compatible with Π, $x^*_\alpha \notin F$. Suppose that $g(\sigma^*_\alpha) = x^*_\alpha$. Since F is implementable in $k-FTBE$, the profile of strategies σ^*_α cannot be a $k-FTBE$, i.e., $\sigma^*_\alpha \notin B^k(g, \mathscr{M}, s)$. Then, there exist a subset $\widetilde{M} \subseteq N$ such that $| \widetilde{M} | = \widetilde{k} \leq k$, a player $j \in N \setminus \widetilde{M}$, a state s' and a profile of strategies $(\sigma'_j, \sigma'_{\widetilde{M}})$, $\quad$ such that:

$$g(\sigma'_j, \sigma^*_{N\setminus(\widetilde{M}\cup\{j\})} \circ \alpha, \sigma'_{\widetilde{M}}) P^j(E^j(s')) g(\sigma^*_j \circ \alpha, \sigma^*_{N\setminus(\widetilde{M}\cup\{j\})} \circ \alpha, \sigma'_{\widetilde{M}}). \qquad (1)$$

We define a profile of constant strategy $(\overline{\sigma}_j(E^j(t)), \overline{\sigma}_{\widetilde{M}}(E^j(t))) = (\sigma'_j, \sigma'_{\widetilde{M}})$ for all t. Then, we have:

$$g(\overline{\sigma}_j, \sigma^*_{N\setminus(\widetilde{M}\cup\{j\})} \circ \alpha, \overline{\sigma}_{\widetilde{M}}) P^j(E^j(s')) g(\sigma^*_j \circ \alpha, \sigma^*_{N\setminus(\widetilde{M}\cup\{j\})} \circ \alpha, \overline{\sigma}_{\widetilde{M}}). \qquad (2)$$

If $| \widetilde{M} | = \widetilde{k} < k$. Let $\widetilde{M}^k \subseteq N \setminus \{j\}$ such that: $\widetilde{M} \subseteq \widetilde{M}^k, \quad | \widetilde{M}^k | = k$ and $\overline{\sigma}_{\widetilde{M}^k} = ((\overline{\sigma}_i)_{i\in\widetilde{M}}, (\sigma^*_i)_{i\in\widetilde{M}^k\setminus\widetilde{M}})$,
Then the equation (2) becomes:

$$g(\overline{\sigma}_j, \sigma^*_{N\setminus(\widetilde{M}^k\cup\{j\})} \circ \alpha, \overline{\sigma}_{\widetilde{M}^k}) P^j(E^j(s')) g(\sigma^*_j \circ \alpha, \sigma^*_{N\setminus(\widetilde{M}^k\cup\{j\})} \circ \alpha, \overline{\sigma}_{\widetilde{M}^k}). \qquad (3)$$

[3]There is a difference between x^i and x_i. The notation x^i means the chosen allocation by the player i while x_i means the part of the player i.

Since $\sigma^* \in B^k(g, \mathscr{M}, s)$, then each $i \in \widetilde{M}^k \cup \{j\}$ satisfies:

$$g(\sigma^*_i, \sigma^*_{N\setminus(\widetilde{M}^k\cup\{j\})}, \overline{\sigma}_{(\widetilde{M}^k\cup\{j\})\setminus\{i\}})R^i(\alpha(E^i(s)))g(\sigma^*_j, \sigma^*_{N\setminus(\widetilde{M}^k\cup\{j\})}, \overline{\sigma}_{\widetilde{M}^k}). \tag{4}$$

We define
$y_\alpha = g(\overline{\sigma}_j, \sigma^*_{N\setminus(\widetilde{M}^k\cup\{j\})} \circ \alpha, \overline{\sigma}_{\widetilde{M}^k})$, and
$x^j_\alpha = g(\sigma^*_j \circ \alpha, \sigma^*_{N\setminus(\widetilde{M}^k\cup\{j\})} \circ \alpha, \overline{\sigma}_{\widetilde{M}^k})$, the equation (3) becomes: $y_\alpha P^j(E^j(s'))x^j_\alpha$
We define $x^i = g(\sigma^*_i, \sigma^*_{N\setminus(\widetilde{M}^k\cup\{j\})}, \overline{\sigma}_{(\widetilde{M}^k\cup\{j\})\setminus\{i\}})$, $y = g(\sigma^*_j, \sigma^*_{N\setminus(\widetilde{M}^k\cup\{j\})}, \overline{\sigma}_{\widetilde{M}^k})$ and $M = (\widetilde{M}^k \cup \{j\})$, then the equation (4) becomes: $x^i R^i(\alpha(E^i(s)))y \quad \forall i \in M$. From the definition of $B(\sigma^*, k)$, it follows that : $x^i \in g(B(\sigma^*, k))$ for all $i \in M$. By the part (iii) of the definition of implementation (Definition 6), we have $x^i \in F$ for all $i \in M$, and by consequent, F is weakly k-Bayesian monotonic.$\|$

5.2 Sufficiency

In this subsection, we state and prove our second main result. We begin by introducing the following condition,

Definition 8 (k-Bayesian monotonicity ($k - BM$)) *:*
A social choice set F is k- Bayesian monotonic if for all α compatible with Π such that $x \in F$, if $x_\alpha \notin F$, then $\exists M \subset N$ and $y : S \to A$ such that $\mid M \mid \geq k+1$, each $i \in M$ satisfies $xR^i(\alpha^i(E^i(s)))y$ $\forall s \in S$ and at least one player $j \in M$ satisfies $y_\alpha P^j(E^j(s))x_\alpha$.

Observation 1: If social choice set is k- Bayesian monotonic, it is also weakly k- Bayesian monotonic.

Observation 2: If $k = 0$, 0-Bayesian monotonicity is quite simply the classical Bayesian monotonicity which is defined by Postlewaite and Schmeider (1986), and Palfrey and Srivastava (1987). Thus, we conclude that k-Bayesian monotonicity implies Bayesian monotonicity.

Assumption 3 (k-Non-Exclusive Information ($k - NEI$)) *:*
Let $n \geq 3$. $\forall M \subseteq N$, $\quad \mid M \mid \geq k+1$, $\quad \forall i \in M$, $\quad \forall s \in S$, $\quad \bigcap_{j\in M\setminus\{i\}} E^j(s) = \{s\}$.

This condition requires that if there are M agents in the economy such that $M \subseteq N$, $\mid M \mid \geq k+1$, then, each group $M-1$ agents have complete information collectively.

Theorem 2 *:*
Let $n \geq 3$, $\quad k+1 < \frac{n}{2}$ and $\quad F \neq \emptyset$. If a social choice set F satisfies k-BM and k-NEI, then F is implementable in k-FTBE.

where $F \neq \emptyset$ mean that $x \in F \Rightarrow x_i(s) \neq 0$ for all i,s .

a) Definition of the mechanism[4]

If $k = 0$, then our mechanism becomes quite simply the mechanism of Postelwaite and Schmeidler (1986). Thus, we study the case where $k > 0$. We construct the following mechanism which to implement F in $k - FTBE$.

Let $\mathscr{M}^i = \Pi^i \times X \times \mathbb{N}$, where

Π^i: Represent the information of an agent,

X : The set of the all feasible allocation rules,

$\mathbb{N}$: The set of non-negative integer,

and $\mathscr{M} = \mathscr{M}^1 \times \mathscr{M}^2 \times ... \times \mathscr{M}^n$.

The outcome function $g : \mathscr{M} \longrightarrow \mathscr{A}$ is defined by the four following rules:

Rule 1: If for a set M of at least $n-k$ agents agree on a state s such that $\bigcap_{i \in M} E^i = \{s\}$ and announce $(E^i, x, 0)$, such that $x \in F$, then $g(m) = x(s)$.

Rule 2 : If for some set $\overline{M}$ of k faulty players and for some one τ non faulty player, the other $n-k-1$ agents announce $(E(s), x, 0)$ such that $x \in F$, $\bigcap_{i \in M \setminus \{\tau\}} E^i = \{s\}$, where $\mid M \mid= n-k$ and the set of $k+1$ announce $(E(s), y, 1) \neq (E(s), x, 0)$, then:

$$g(m) = \begin{cases} y(s) & if\ xR^{i'}(\alpha(E^{i'}(s)))y,\ \forall i' \in (\overline{M} \cup \{\tau\}),\ \forall s \in S,\ s.t.\ \bigcap_{i \in M \setminus \{\tau\}} E^i = \{s\}, \\ x(s) & if\ yP^{i'}(\alpha(E^{i'}(s)))x,\ \exists i' \in (\overline{M} \cup \{\tau\}),\ \exists s \in S,\ s.t.\ \bigcap_{i \in M \setminus \{\tau\}} E^i = \{s\}. \end{cases}$$

Rule 3 : 3-a): If at most $n-k$ agents are in disagreement about the same sate such that $\bigcap_{i \in M} E^i = \emptyset$ or $\exists i \in M$ with $\mid M \mid \leq n-k$, such that $n \neq 0$ or $x^i \neq x$, then: $g(m) = 0$.

3-b): If for all sets constituted of k faulty agents and of a non-faulty agent, says τ, the other $n-k-1$ agents are in disagreement about the same state, i.e., there exists at least an agent $\hat{\tau} \in \tilde{N} \setminus \{\tau\}$ who does not agree on the state, therefore $(\bigcap_{i \in M \setminus (\tau \cup \hat{\tau}\}} E^i(t)) \bigcap E^{\hat{\tau}} = \emptyset$ with $\mid M \mid= n-k$, or, $(x^{\hat{\tau}}, n^{\hat{\tau}}) \neq (x, o)$, then: $g(m) = 0$.

[4]Our mechanism is adapted to those of Postelwaite and Schmeidler (1986), Palfrey and Srivastava (1989) in the incomplete information environments and to that of Eliaz (2002) in complete information.

Rule 4 : In the all other cases,

$$g^i(m) = \begin{cases} x^{j^*}(s) & \text{if} \quad n^{j^*} \geq n^i \quad \forall i \in N, \\ 0 & otherwaise. \end{cases}$$

where $x^{j^*} = \frac{\overline{w}}{l}$, such that l is number of the players having the greatest integer.

b) Explanation of the mechanism

In the above mechanism, each message of an agent contains: a report concerning its own event [5], an allocation rule and an integer number.

Initially, in rule 1, if the reports of the non-faulty players agree on a state s and if these players request the same allocation rule and zero, then we would prefer that this rule is the outcome independently of the behavior of the faulty players. Since we have at least a group of $n-k$ non-faulty players, rule 1 guarantees the determination of the outcome.

The rule 2 studies how us should proceed if there exist exactly a set of $k+1$ players which deviates and sends a message which contains $y \in F$ and the number "1", whereas the others ($N-k-1$ players) request the same allocation $x \in F$, an event and zero. If this last group which constitutes the majority agrees on a allocation rule, the outcome should be to determine by the group of $k+1$ players (the minority) only if it is truthful. This condition is met only if the minority prefers x at its own report for all its possible types, it obtains its will and the result will be $y(s)$. In the contrary case, the choice of minority must be neglected.

The rule 3 studies two situations. The first, when at least $n-k$ players are in disagreement on the same state. second situation, when for any set of $k+1$ players, the others $n-k-1$ are in disagreement on the same state. In these situations, there are some messages of disagreement where the planner cannot detect the player who "causes" this disagreement, thus, each player is severely punished and it receives zero.

When the majority of the players are in disagreement among them selves, there is no possibility of checking which is truthful and which is not. Consequently, we would like to prevent the strategies of disagreement which bring undesirable equilibrium outcomes. whenever the majority choose different strategies, the rule 4 guarantees that there is one player with an incentive to deviate. This player reports the highest integer and receives all the resources.

[5] As in the case of the mechanism in complete information, the players who report an event as if they report a subset of possible preference profiles, thus, all players are invited to indicate that they know about the others, as well as they know about them selves.

Lemma 1 *:*
If a profile of strategies σ is a $k-FTBE$ for the mechanism Γ, then, $\forall s \in S$, the equilibrium outcomes $g(\sigma(E(s)))$ come from rule 1.

Proof. Let σ be a $k - FTBE$ for the mechanism Γ. Let us show that $\forall s \in S$, the equilibrium outcomes $g(\sigma(E(s)))$ come from rule 1. Suppose not. Therefore, $\exists s \in S$ such that the equilibrium outcomes $g(\sigma(E(s)))$ do not come from rule 1. Then, there exists three cases to be considered:

Case 1. $\exists s \in S$ and there is a set of $k+1$ players such that the equilibrium outcomes $g(\sigma(E(s)))$ come from rule 2. In this case the $n-k-1$ players are an agreement on same triple $(E(s), x, 0)$ and the set of $k+1$ (minority) announce different triple $(E(s), y, 1)$ such that $y \neq x$.

Since $n \geq 3$ and $k+1 < \frac{n}{2}$, $\exists j \in N \setminus (\overline{M} \cup \{\tau\})$ where $\tau \in \tilde{N}$, such that: $g^j(\sigma(E(s))) < \bar{w}$. (1)

But, the player j can move from rule 2 to rule 4 by changing its integer to higher integer. We define for this player the vector:

$$\tilde{\sigma}^j(E^j(t)) = \begin{cases} (E^j(s), x, n^j) & \text{si} \quad t \in E^j(s), \\ \sigma^j(E^j(t)) & \text{si} \quad t \notin E^j(s). \end{cases}$$

where $n^j > n^i$, $\forall i \neq j$, $\forall t \in S$. We define also $\tilde{\sigma}(E(s)) = (\sigma^{-j}, \tilde{\sigma}^j)(E(s))$. Therefore, the outcomes $g(\tilde{\sigma}(E(s)))$ come from rule 4. Thus, $g^j(\tilde{\sigma}(E(s))) = x^{j^*} = \bar{w}$. From inequality (1), we have: $g^j(\tilde{\sigma}(E(s))) > g^j(\sigma(E(s)))$.

Now, we consider all $t \in E^j(s)$ and we show that $g^j(\tilde{\sigma}(E(t))) \geq g^j(\sigma(E(t)))$.

α) If $g(\sigma(E(t)))$ comes from rule 1 and if we suppose that $x \in F$ for at least $n-k$ players, then $g^j(\sigma(E(t)) = x^j(t)$. Therefore, $g(\tilde{\sigma}(E(t)))$ comes from rule 4, thus, $g^j(\tilde{\sigma}(E(t)) = x^{j^*}(t) \geq g^j(\sigma(E(t))$, in this case j receives $\bar{w}$ and there is at least as well off.

β) If $g(\sigma(E(t)))$ comes from rule 2 where there exists a minority of players constituted of k-faulty players and one non-faulty player, says τ, then the equilibrium outcomes $g(\tilde{\sigma}(E(t)))$ come from rule 4 because $n^j > n^i$ for all $i \in N$, in this case j obtains $\bar{w}$ and there is at least as well off.

γ) If $g(\sigma(E(t)))$ comes from rule 3, then $g^j(\sigma(E(t)) = 0$, thus, j can not possibly be worse off.

δ) If $g(\sigma(E(t)))$ come from rule 4, then $g(\tilde{\sigma}(E(t)))$ comes from rule 4, in this case j obtains $\bar{w}$ and there is at least as well off.

Therefore, if for some s, $g(\sigma(E(s)))$ comes rule 2, σ cannot be an equilibrium, a contradiction.

Case 2. $\exists s \in S$ such that the equilibrium outcomes $g(\sigma(E(s)))$ come from rule 3. In this case, there exists j such that $g^j(\sigma(E(s)) = 0$. We define

$\tilde{\sigma}^j$ and $\tilde{\sigma} = (\sigma^{-j}, \tilde{\sigma}^j)$ as in case 1. We can show easily that the player j is strictly better off by using $\tilde{\sigma}^j$.

Case 3. $\exists s \in S$ such that the equilibrium outcomes $\sigma(E(s))$ come from rule 4. We define $\tilde{\sigma}^j$ and $\tilde{\sigma} = (\sigma^{-j}, \tilde{\sigma}^j)$ of the same way for some player j and we show that j is strictly better off by using $\tilde{\sigma}^j$.

Thus, σ cannot be a equilibrium.

c) Proof of the theorem.

Step 1: We show that $\forall x \in F$, the strategies vector $\sigma_i^* = (E^i(s), x, 0)$, $\forall i \in N$ and $\forall s \in S$ is a $k-FTBE$.

Let $x \in F$ for some $E(s) \in \Pi$. Suppose that all players use the strategies vector $\sigma_i^* = (E^i(s), x, 0)$ for all $i \in N$ and for all $s \in S$. Let us show that the profile de strategy $\sigma^* = (\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ is a $k-FTBE$ of game (g, S), i.e., $\forall M \subseteq N$ such that $\mid M \mid \leq k$, $\quad \forall \sigma_i : \Pi^i \rightarrow \mathscr{M}^i$ and $\forall \sigma_M : \Pi^M \rightarrow \mathscr{M}^M$, $g(\sigma_i^*, \sigma^*_{N\backslash\{M\}\cup\{i\}}, \sigma_M) R^i(E^i(s)) g(\sigma_i, \sigma^*_{N\backslash\{M\}\cup\{i\}}, \sigma_M)$.
To verified that σ^* is effectively a $k-FTBE$ of $\mathscr{M}$, we consider an unilateral deviation by non-faulty player, i.e., $\exists M \subseteq N$ such that $\mid M \mid \leq k$, $\exists \tau \in N$, $\exists \sigma_\tau : \Pi^\tau \rightarrow \mathscr{M}^\tau$ and $\exists \sigma_M : \Pi^M \rightarrow \mathscr{M}^M$, such that:$g(\sigma_\tau, \sigma^*_{N\backslash\{M\}\cup\{\tau\}}, \sigma_M) P^\tau(E^\tau(s)) g(\sigma_\tau^*, \sigma^*_{N\backslash\{M\}\cup\{\tau\}}, \sigma_M)$. We show that this deviation is not profitable.
We define the deviation of player τ at state s by: $\sigma_\tau(E^\tau(s)) = (E^\tau(s), x', n') \neq (E^\tau(s), x, 0)$. There is three cases to consider:

Case 1: For all $i \in N\backslash M$ with $\mid N\backslash M \mid = n-k$, if $x' \neq x$, $n' = 1$ and $\bigcap_{i \in N\backslash(M\cup\{\tau\})} E^i \neq \emptyset$, the equilibrium outcomes $g(\sigma_\tau, \sigma^*_{N\backslash M\cup\{\tau\}}, \sigma_M)$ come from rule 2. Thus, if for all $i \in (M\cup\{\tau\})$, for all $s \in S$, $xR^i(\alpha(E^i(s)))y$, then, by rule 2, $g(\sigma_\tau, \sigma^*_{N\backslash M\cup\{\tau\}}, \sigma_M) = y$. Therefore, $g(\sigma^*) R^i g(\sigma_\tau, \sigma^*_{N\backslash M\cup\{\tau\}}, \sigma_M)$. If there is $i \in (M \cup \{\tau\})$ and $s \in S$ such that $yP^i(\alpha(E^i(s)))x$, then, by rule 2, $g(\sigma_\tau, \sigma^*_{N\backslash M\cup\{\tau\}}, \sigma_M) = x$. Therefore, $g(\sigma^*) I^i g(\sigma_\tau, \sigma^*_{N\backslash M\cup\{\tau\}}, \sigma_M)$, (by the $k-NEI$ condition, the incentive constraints are not posed in pure exchange economies).

Case 2: For all $i \in N\backslash M$ with $\mid N\backslash M \mid = n-k$, if $(x', n') \neq (x, 0)$ or $\bigcap_{i \in N\backslash(M\cup\{\tau\})} E^i = \emptyset$, the equilibrium outcomes $g(\sigma_\tau, \sigma^*_{N\backslash M\cup\{\tau\}}, \sigma_M)$ come from rule $3-b$. Therefore, $g(\sigma_\tau, \sigma^*_{N\backslash M\cup\{\tau\}}, \sigma_M) = 0$ and the player τ receives 0. Thus, the player τ submit $(E^\tau(s), x, 0)$ instead of $(E^\tau(s), x', n')$.

Case 3: For all $i \in N\backslash M$ with $\mid N\backslash M \mid = n-k$, if $\bigcap_{i \in N\backslash M} E^i = \emptyset$, then, we apply rule (3-a), the player τ receives 0. Thus, the player τ submit $(E^\tau(s), x, 0)$ instead of $(E^\tau(s), x', n')$.

From cases 1, 2 and 3, we conclude that σ^* is a $k - FTBE$.

Step 2: We show that if σ^* is a $k - FTBE$, then:
1) $x^* = g(\sigma^*) \in F$.
2) $\forall \sigma \in B(\sigma^*, k), \quad g(\sigma) \in F$.

Sub-step 2.1: We show that if σ^* is a $k - FTBE$, then $x^* = g(\sigma^*) \in F$.

Let σ^* be a $k - FTBE$ for the mechanism Γ such that σ^* is defined by $\sigma_i^*(E^i(s)) = (E^i(s), x^*, 0)$ for at least $n - k$ players and for all $s \in S$. By Lemma 1, $\forall s \in S$, the equilibrium outcome $g(\sigma^*)$ come from rule 1. Since F satisfies closure, then $x^* \in F$.

We consider $\alpha_i^*(E^i(s))$ for all $i \in N$ and for all $s \in S$ such that α^* is a deception. Then by definition of mechanism, $g(\sigma^*) = x^*_{\alpha^*}$. We show that $g(\sigma^*) = x^*_{\alpha^*} \in F$.
Suppose not. Therefore, $x^*_{\alpha^*} \notin F$. Since F is k-Bayesian monotonic, $\exists M \subset N, s \in S$ and $y : S \to A$ such that $\mid M \mid \geq k + 1$, who satisfies $x^* R^i(\alpha_i^*(E^i(s)))y$ for all $i \in M$ and $\exists j \in M$ such that $y_{\alpha^*} P^j(E^j(s))x^*_{\alpha^*}$. Then, by using the strategy σ_j^*, player j does not have better response when observes $E^j(s)$. Thus, if k player of subset $M \setminus \{j\}$ use the strategy $\sigma' : \Pi \to \mathscr{M}$ with $\sigma'(\alpha^*(E(s))) = (\alpha^*(E(s)), y, 1)$ and the remainder of players use the strategy σ^*, then if at $E^j(s)$, player j reports $(\alpha_j^*(E^j(s)), y, 1)$, it will move from 1 to rule 2 and it will have the outcome y_{α^*}. This contradicted our condition which is $\sigma^* \in k - FTBE$. Then $x^*_{\alpha^*} \in F$.

Sub-step 2.2: We show that if σ^* is a $k - FTBE$, then $\forall \sigma \in B(\sigma^*, k), \quad g(\sigma) \in F$.
Since σ^* is a $k - FTBE$ for mechanism Γ, then from lemma 1, $\forall s \in S$, the equilibrium outcomes $g(\sigma^*)$ come from rule 1. Therefore, we can represent the strategy vector σ^* in the following way:
$\sigma^* = ((E^1, x^*, 0), (E^2, x^*, 0), ..., (E^j, x^*, 0), ..., (E^{n-k}, x^*, 0), (E^1, x^1, m^1), ...,$
$(E^h, x^h, m^h), ..., (E^k, x^k, m^k))$ and $x^* = g(\sigma^*) \in F$. Let $M \subset N$ be a subset of k players. Let $\sigma \in B(\sigma^*, k)$, if
$\sigma = ((E^1, x^*, 0), (E^2, x^*, 0), ..., (E^j, x^*, 0), ..., (E^{n-k}, x^*, 0), (E'^1, x'^1, m'^1), ...,$
$(E'^h, x'^h, m'^h), ..., (E'^k, x'^k, m'^k))$, then in this case, the members of subset M deviate in k positions and them equilibrium outcomes $g(\sigma)$ come from rule 1, i.e., $g(\sigma) = x^* \in F$.
Suppose not. The strategy vector σ^* is a $k - FTBE$ and $\exists \sigma \in B(\sigma^*, k), \quad g(\sigma) \notin F$, or quite simply σ^* is a $k - FTBE$ and $\exists \sigma \in B(\sigma^*, k)$ such that the equilibrium outcome $g(\sigma)$ does not come from rule 1, then, there exists three

cases to considerer:

Case 1: $\exists s \in S$ such that the outcome $g(\sigma)$ comes from rule 4 of mechanism Γ and $g(\sigma) \neq x^*$. Suppose that $\exists j \in N \backslash M$ such that $g^j(\sigma^*) < \overline{w}$, $\exists h \in M$ and $M \subseteq N \backslash \{j\}$. Let
$\widetilde{\sigma} = ((E^1, x^*, 0), (E^2, x^*, 0), ..., (\widetilde{E}^j, \widetilde{x}^j, \widetilde{m}^j), ..., (E^{n-k}, x^*, 0), (\widetilde{E}^1, \widetilde{x}^1, \widetilde{m}^1), ...,$ $(E'^h, x'^h, m'^h), ..., (\widetilde{E}^k, \widetilde{x}^k, \widetilde{m}^k))$, profile of strategies in which members of subset $(\{j\} \cup (M \backslash h))$ deviate in k positions such that player j announce triple $(E^j, \widetilde{x}^j, \widetilde{m}^j)$ where $\widetilde{m}^j > m^i$ $\forall i \in N \backslash \{j\}$ and player h keeps his triple (E'^h, x'^h, m'^h). Thus, $\widetilde{\sigma} \in B(\sigma^*, k)$ and by rule 4, $g^j(\widetilde{\sigma}) = \overline{w} > g^j(\sigma^*)$ at state s.

Now, we consider all $t \in E^j(s)$ and we show that $g^j(\widetilde{\sigma}) \geq g^j(\sigma^*)$. There is four subcases to consider:

Subcase 1: If the outcome $g(\sigma(E(t)))$ comes from rule 1, then $g(\widetilde{\sigma}(E(t)))$ comes from rule 4. Therefore, $g(\widetilde{\sigma}^j(E(t))) = \overline{w} \geq g^j(\sigma^*(E(t)))$.

Subcase 2: If the outcome $g(\sigma(E(t)))$ comes from rule 2, then $\exists M' \subset M$ of k' players, with $k' < k$ et $\exists \tau \in N \backslash M'$ such that the subset $(M' \cup \{\tau\})$ announce triple $(E', y, 1) \neq (E, x^*, 0)$, with $y = x'^1 = ... = x'^h = ... = x'^k$ and $1 = m'^1 = ... = m'^h = ... = m'^k$. Therefore, for $j \in N \backslash M'$ who announce triple $(E^j, \widetilde{x}^j, \widetilde{m}^j)$ where $\widetilde{m}^j > m^i$ $\forall i \in N \backslash \{j\}$, if $j = \tau$, then $g(\widetilde{\sigma}(E(t)))$ comes from rule 2 and $g^j(\widetilde{\sigma}(E(t))) = g^j(\sigma^*(E(t)))$.

If $j \neq \tau$, then:

$$g^j(\sigma^*(E(t))) = \begin{cases} y^j(t) & if\ xR^{i'}(\alpha(E^{i'}(t)))y,\ \forall i' \in (M' \cup \{\tau\}),\ \forall t \in S,\ t.q.\ \bigcap_{i \in N \backslash (M' \cup \{\tau\})} E^i = \{t\}, \\ x^j(t) & if\ yP^{i'}(\alpha(E^{i'}(t)))x,\ \exists i' \in (M' \cup \{\tau\}),\ \exists t \in S,\ t.q.\ \bigcap_{i \in N \backslash (M' \cup \{\tau\})} E^i = \{t\}. \end{cases}$$

In this case $g(\widetilde{\sigma}(E(t)))$ comes from rule 4 and $g^j(\widetilde{\sigma}(E(t))) = \overline{w}$, by feasibility $g^j(\widetilde{\sigma}(E(t))) \geq y^j(t)$ or $g^j(\widetilde{\sigma}(E(t))) \geq x^j(t)$, therefore $g^j(\widetilde{\sigma}(E(t))) \geq g^j(\sigma^*(E(t)))$.

Subcase 3: If the outcome $g(\sigma(E(t)))$ comes from rule 3, then $g^j(\sigma(E(t))) = 0$. Thus, player j cannot have a negative result.

Subcase 4: If the outcome $g(\sigma(E(t)))$ come from 4, then $g(\widetilde{\sigma}(E(t)))$ come from rule 4. Thus, $g(\widetilde{\sigma}(E(t))) \geq g(\sigma^*(E(t)))$.

Therefore, in case, $g(\widetilde{\sigma}(E(s)))P^j(E(s))g(\sigma^*(E(s)))$, a contradiction, because σ^* is a $k - FTBE$.

Case 2: $\exists s \in S$ such that the outcome $g(\sigma)$ come from rule 3 of mechanism Γ and $g(\sigma) \neq x^*$, in this case $g(\sigma) = 0$. We have $\sigma_i = \sigma_i^*$ $\forall i \in N \backslash M$. Thus, for a player $j \in N \backslash M$, $g^j(\sigma*) = 0$. Suppose that $\exists h \in M$

and $M \subseteq N \backslash \{j\}$.
Let $\widetilde{\sigma} = ((E^1, x^*, 0), (E^2, x^*, 0), ..., (\widetilde{E}^j, \widetilde{x}^j, \widetilde{m}^j), ..., (E^{n-k}, x^*, 0), (\widetilde{E}^1, \widetilde{x}^1, \widetilde{m}^1), ..., (E'^h, x'^h, m'^h), ..., (\widetilde{E}^k, \widetilde{x}^k, \widetilde{m}^k))$, profile of strategies in which the members of subset $(\{j\} \cup (M\backslash h))$ deviate in k positions such that player j announce triple $(E^j, \widetilde{x}^j, \widetilde{m}^j)$ where $\widetilde{m}^j > m^i\ \forall i \in N\backslash\{j\}$, player h keeps his triple (E'^h, x'^h, m'^h). Thus $\widetilde{\sigma} \in B(\sigma^*, k)$ and by rule 4, $g^j(\widetilde{\sigma}) = \overline{w} > g^j(\sigma^*)$ at state s.

We consider all $t \in E^j(s)$. We show in the same way as in case 1 that $g^j(\widetilde{\sigma}) \geq g^j(\sigma^*)$ considering four subcases.

Therefore, in case 2, $g(\widetilde{\sigma}(E(s)))P^j(E(s))g(\sigma^*(E(s)))$, a contradiction, because σ^* is a $k - FTBE$.

Case 3: $\exists s \in S$ such that the outcome $g(\sigma)$ come from rule 2 of mechanism Γ and $g(\sigma) \neq x^*$, there exists $M' \subset M$ of k' players, with $k' < k$ and $\exists \tau \in N\backslash M'$ such that the subset $(M' \cup \{\tau\})$ announce triple $(E', y, 1) \neq (E, x^*, 0)$, with $y = x'^1 = ... = x'^h = ... = x'^k$ and $1 = m'^1 = ... = m'^h = ... = m'^k$. Soit $\widetilde{\sigma}$ a profile of strategies in which some player $j \in N\backslash(M' \cup \{\tau\})$ announce triple $(E^j, \widetilde{x}^j, \widetilde{m}^j)$ where $\widetilde{m}^j > m^i\ \forall i \in N\backslash\{j\}$. Let $h \in M'$ a player who keeps his triple (E'^h, x'^h, m'^h). Thus, the subset of players $(\{j\} \cup M'\backslash\{h\})$ deviate in k' positions. Therefore, $\widetilde{\sigma} \in B(\sigma^*, k)$. We have $\sigma_i = \sigma_i^*\ \forall i \in N\backslash(M' \cup \{\tau\})$ and $\widetilde{\sigma}_i = \sigma_i\ \forall i \in N\backslash(\{j\} \cup M'\backslash\{h\})$. By rule 4, $g^j(\widetilde{\sigma}) = \overline{w} > g^j(\sigma^*)$ at state s.

As in case 1, for all $t \in E^j(s)$, we need to show that $g^j(\widetilde{\sigma}) \geq g^j(\sigma^*)$.

Therefore, in case 2, $g(\widetilde{\sigma}(E(s)))P^j(E(s))g(\sigma^*(E(s)))$, a contradiction, because σ^* is a $k - FTBE$. $\|$

5.2.1 Applications of $k - FTBE$ implementability

Application 1. An allocation $x^* : S \to A$ is interim no-envy (IN_e) if for all i, j, s,
$\sum_{t \in E^i(s)} p^i(j \neq i \in \widetilde{N} | \widetilde{N}, i \in \widetilde{N}) q^i(t \mid E^i(s)) U^i(x_i^*(t), t) \geq$
$\sum_{t \in E^i(s)} p^i(j \neq i \in \widetilde{N} | \widetilde{N}, i \in \widetilde{N}) q^i(t \mid E^i(s)) U^i(x_j^*(t), t)$.

Proposition 1 .
Let $k < n$. The interim no-envy IN_e satisfies k-Bayesian monotonicity.

Proof. For $k < n$, we show that IN_e satisfies k-Bayesian monotonicity. Let α a deception compatible with Π. Let $x^* \in IN_e$ but $x_\alpha^* \notin IN_e$.
$x_\alpha^* \notin IN_e$ implies that there exists i, j, s such that
$\sum_{t \in E^i(s)} p^i(j \neq i \in \widetilde{N} | \widetilde{N}, i \in \widetilde{N}) q^i(t \mid E^i(s)) U^i(x_{j\alpha}^*(t), t) >$

$\sum_{t\in E^i(s)} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t \mid E^i(s))U^i(x^*_{i\alpha}(t), t)$.

Consider the allocation y for all $t \in S$,

$$y_h(t) = \begin{cases} x^*_h(t) & if \quad h \neq i, j \\ x^*_j(t) & if \quad h = i \qquad (1), \\ x^*_i(t) & if \quad h = j \end{cases}$$

It is clear that $y \in \mathscr{A}$. Thus, we have by construction
$\sum_{t\in E^i(s)} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t \mid E^i(s))U^i(y_{i\alpha}(t), t) >$
$\sum_{t\in E^i(s)} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t \mid E^i(s))U^i(x^*_{i\alpha}(t), t)$. (2)

It remains to show that for $k < n$, there exists $M \subseteq N$ with $\mid M \mid \geq k+1$ such that $M \supseteq \{i\}$ and the expected utility for all player i of set M for allocation x^*_i at state t' is greater than the expected utility for some given allocation y_i at state t'. We have $x^* \in IN_e$. Thus, by definition, for all i, j, s,
$\sum_{t'\in E^i(s')} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t' \mid E^i(s'))U^i(x^*_i(t'), t') \geq$
$\sum_{t'\in E^i(s')} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t' \mid E^i(s'))U^i(x^*_j(t'), t')$. (3)

From (1), we have for all i, j, s',
$\sum_{t'\in E^i(s')} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t' \mid E^i(s'))U^i(x^*_i(t'), t') \geq$
$\sum_{t'\in E^i(s')} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t' \mid E^i(s'))U^i(y_i(t'), t')$. (4)

Thus, by taking $M \equiv N$, we complete the proof. $\|$

Application 2. Let $\beta^* : S \rightarrow \Re^l_+$ be a price function such that $\beta^* \neq 0$. Let x^* be an allocation rule such that $x^* \in \mathscr{A}$. Let w_i be the endowment of each player $i \in N$ with $w_i > 0$. The pair (β^*, x^*) is a Constrained Rational Expectation Equilibrium (CREE) if:

1. $\beta^*(s)x^*_i \leq \beta^*(s)w_i(s)$ for all $s \in S$ and for all $i \in N$,

2. Measurability of allocation with respect to prices: For each agent i and each $t \in E^i(s)$, $\beta^*(t) = \beta^*(s) \Rightarrow x^*_i(t) = x^*_i(s)$,

3. $\sum\limits_{t\in E^i(s)} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t \mid E^i(s))U^i(x^*_i(t), t) \geq$
$\sum\limits_{t\in E^i(s)} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t \mid E^i(s))U^i(y_i(t), t)$ for all $s \in S$, for all $i \in N$ and all $y_i(s)$ such that $y_i(s) \leq \overline{w}$ and $\beta^*(s)y_i(s) \leq \beta^*(s)w_i$,

4. $\sum_{i\in N} x^*_i(s) = \overline{w}$ for each $s \in S$.

Proposition 2 .
Let $0 < k < n$. The CREE social choice set satisfies k-Bayesian monotonicity.

Proof. For $0 < k < n$, we show that $CREE$ satisfies k- Bayesian monotonicity. Let α a deception compatible with Π. Given some $s \in S$, let $s' = \alpha(s)$. Let $x^* = (x_1^*, x_2^*, ..., x_i^*, ..., x_n^*) \in X$ such that x^* is an $CREE$ but x_α^* is not. Thus, there exists j, s, and an allocation z_j with $z_j \leq \overline{w}$ such that $\beta^*(\alpha(s))z_j \leq \beta^*(\alpha(s))w_j(\alpha(s))$ and
$\sum_{t \in E^i(s)} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t \mid E^i(s))U^i(z_j, t) >$
$\sum_{t \in E^i(s)} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t \mid E^i(s))U^i(x_{j\alpha}^*(t), t)$. (5)

Define y_j as follows:

$$y_j(t') = \begin{cases} z_j & \text{if } \ t' \in E^i(s'), \\ x_j^*(t') & \textit{otherwise.} \end{cases}$$

For all $t \in E^i(s)$, $y_{j\alpha}(t) = z_j$. Thus, inequality (5) implies that
$\sum_{t \in E^i(s)} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t \mid E^i(s))U^i(y_{j\alpha}(t), t) >$
$\sum_{t \in E^i(s)} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t \mid E^i(s))U^i(x_{j\alpha}^*(t), t)$.

It remains to show that for $k < n$, there exists $M \subseteq N$ with $\mid M \mid \geq k+1$ such that $M \supseteq \{j\}$ and the expected utility for all player i of set M for allocation x_i^* at state t' is greater than the expected utility for some given allocation y_i at state t'.

Let $y_i(t') = \frac{\overline{w} - y_j(t')}{n-1}$ for all $i \neq j$. It is clair that $y(t') \in \mathscr{A}$ for all t'. We have x^* is an $CREE$, thus for all i, s', and for all $y_i(s') \leq \overline{w}$ such that $\beta^*(s')y_i(s') \leq \beta^*(s')w_i$,
$\sum_{t' \in E^i(s')} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t' \mid E^i(s'))U^i(x_i^*(t'), t') \geq$
$\sum_{t' \in E^i(s')} p^i(j \neq i \in \widetilde{N}|\widetilde{N}, i \in \widetilde{N})q^i(t' \mid E^i(s'))U^i(y_i(t'), t')$.

Thus, by taking $M \equiv N$, we complete the proof.$\|$

In NEI information structures, the interim no-envy IN_e and the CREE social choice set satisfy k-Bayesian monotonicity. Therefore, they are $k - FTBE$ implementable by theorem 2. Thus, we have the following corollary.

Corollary 1 *: Let $n \geq 3$ and $k + 1 < \frac{n}{2}$. The interim no-envy IN_e and the CREE social choice set are $k - FTBE$ implementable by theorem 2.*

6 Example and discussion

In following example, we illustrate that there are weakly k-Bayesian monotonic social choice set which are not Bayesian monotonic. [6]

Example 1. $N = \{1,2,3\}$, $k = 1$, $(p^i = \frac{1}{3})_{i=1,2,3}$, $(q^i = \frac{1}{2})_{i=1,2,3}$, $S = \{s_1, s_2, s_3, s_4\}$. There is one good. The endowment of each agent is one for each state of nature. The agents' partition are:
$\Pi^1 = \big\{\{s_1, s_2\}, \{s_3, s_4\}\big\}$,
$\Pi^2 = \big\{\{s_2, s_3\}, \{s_1, s_4\}\big\}$,
$\Pi^3 = \big\{\{s_1, s_3\}, \{s_2, s_4\}\big\}$,

The utility functions are as given in the table below

	s_1	s_2	s_3	s_4
1	$6x$	$6x$	$3x$	$3x$
2	$3x$	$6x$	$6x$	$3x$
3	$6x$	$3x$	$6x$	$3x$

Consider the social choice functions $\widetilde{x}$, $\widetilde{y}$ et $\widetilde{z}$ given in the next tables.

$\widetilde{x}$	s_1	s_2	s_3	s_4
1	1.5	1	1	0.5
2	0.5	1	1.25	1.25
3	1	1	0.75	1.25

$\widetilde{y}$	s_1	s_2	s_3	s_4
1	1	1	1.5	0.5
2	1	1.5	1	0.5
3	1	0.5	0.5	2

$\widetilde{z}$	s_1	s_2	s_3	s_4
1	1.5	1.5	1	0
2	1	1.35	1.35	0.30
3	0.5	0.15	0.65	2.70

Consider a deception α with $\alpha(s_i) = s_4$, $i = 1, ..., 4$. It is clear that α compatible with Π. Consider player 1, 2, and 3 observe respectively events $\{s_1, s_2\}$, $\{s_2, s_3\}$ and $\{s_1, s_3\}$. By simple calculation of utility for each player, we obtain the following tables.

$\alpha(E^i(s))$:	1	2	3
	$\widetilde{z} \leftarrow 3.00$	$\widetilde{z} \leftarrow 2.70$	$\widetilde{x} \leftarrow 1.75$
	$\widetilde{x} \leftarrow 2.50$	$\widetilde{y} \leftarrow 2.50$	$\widetilde{y} \leftarrow 1.50$
	$\widetilde{y} \leftarrow 2.00$	$\widetilde{x} \leftarrow 2.25$	$\widetilde{z} \leftarrow 1.15$

$E^i(s)$:	1	2	3
	$\widetilde{y}_\alpha \leftarrow 1.000$	$\widetilde{y}_\alpha \leftarrow 0.750$	$\widetilde{z}_\alpha \leftarrow 1.425$
	$\widetilde{x}_\alpha \leftarrow 0.750$	$\widetilde{z}_\alpha \leftarrow 0.650$	$\widetilde{y}_\alpha \leftarrow 1.250$
	$\widetilde{z}_\alpha \leftarrow 0.500$	$\widetilde{x}_\alpha \leftarrow 0.625$	$\widetilde{x}_\alpha \leftarrow 1.125$

Consider the social choice set F on $\Re$ as: $F = \{\widetilde{x}_\alpha, \widetilde{y}_\alpha, \widetilde{z}\}$.

[6] A social choice set F is Bayesian monotonic if for all α compatible with Π such that $x \in F$, if $x_\alpha \notin F$, then $\exists y : S \to A$ such that each $i \in N$ satisfies $xR^i(\alpha^i(E^i(s)))y$ $\forall s \in S$ and at least one player $j \in N$ satisfies $y_\alpha P^j(E^j(s))x_\alpha$.

In this example, we have $\widetilde{z} \in F$ and $\widetilde{z}_\alpha \notin F$. For subset $M = \{1, 2\}$, the players 1 and 2 preferred $\widetilde{z}$ to any other allocation rule in $\alpha(E^i(s))$. However, in $E^i(s)$ at least one these players reverse his preference by ranking $\widetilde{y}_\alpha$ at the top. Thus, the social choice set F satisfies weak k-Bayesian monotonicity.

Now, if all players are non-faulty, then $k = 0$ and $(p^i = 1)_{i=1,2,3}$. In this case, we have the following tables.

$\alpha(E^i(s))$:	1	2	3
	$\widetilde{z} \leftarrow 9.00$	$\widetilde{z} \leftarrow 8.10$	$\widetilde{x} \leftarrow 5.25$
	$\widetilde{x} \leftarrow 7.50$	$\widetilde{y} \leftarrow 7.50$	$\widetilde{y} \leftarrow 4.50$
	$\widetilde{y} \leftarrow 6.00$	$\widetilde{x} \leftarrow 6.75$	$\widetilde{z} \leftarrow 3.45$

$E^i(s)$:	1	2	3
	$\widetilde{y}_\alpha \leftarrow 3.000$	$\widetilde{y}_\alpha \leftarrow 2.250$	$\widetilde{z}_\alpha \leftarrow 4.275$
	$\widetilde{x}_\alpha \leftarrow 2.250$	$\widetilde{z}_\alpha \leftarrow 1.950$	$\widetilde{y}_\alpha \leftarrow 3.750$
	$\widetilde{z}_\alpha \leftarrow 1.500$	$\widetilde{x}_\alpha \leftarrow 1.875$	$\widetilde{x}_\alpha \leftarrow 3.375$

Since the ranks of allocation rules do not change, we have the same social choice set $F = \{\widetilde{x}_\alpha, \widetilde{y}_\alpha, \widetilde{z}\}$. We have $\widetilde{x}_\alpha \in F$. For players 1 and 2, the allocation rules $\widetilde{x}_\alpha$ and $\widetilde{x}$ have the same ranks in $E^i(s)$ and $\alpha(E^i(s))$. For player 3, the allocation rule $\widetilde{x}_\alpha$ ranked at the bottom in $E^i(s)$ and the allocation rule $\widetilde{x}$ ranked at the top in $\alpha(E^i(s))$, but $\widetilde{x} \notin F$. Therefore F does not satisfy Bayesian monotonicity.

We conclude that there exists the social choice sets which are not Bayesian monotonic, but they are weak k-Bayesian monotonic. By the same reasoning of example 1, we can also show there exists the Bayesian monotonic social choice sets wich are not weakly k-Bayesian monotonic. Thus, there is no logical relationship between Bayesian monotonicity and weak k- Bayesian monotonicity.

7 Conclusion

We have extended the concept of Fault Tolerant Implementation of Eliaz (2002) to Bayesian approach. We have characterized Bayesian implementable SCS's in pure exchange economic environments with non-exclusive information when there exists at least k faulty players in the population. Firstly, we have defined new notions of equilibrium and implementation. Secondly, we have proved that an extended version of weak k-monotonicity, called weak k-Bayesian monotonicity, is necessary for implementation. Also, we have showed that an extended version of k-monotonicity, called k-Bayesian monotonicity together with $k - NEI$ condition is sufficient for implementation.

In our work, in order to simplify, we have used non-exclusive information. But, there remain others open questions. For example, first extension not addressed in the paper is the fault tolerant Bayesian implementation with exclusive information. Second extension not analyzed in this work is our notion of implementation in more general environments than the pure exchange setting examined here.

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